Note on the damping of free-surface oscillations due to drainage

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It is shown that the draining of liquid from a vertical tank produces a small but positive damping of free-surface oscillations. This conclusion is consistent with the experimental observations of Brooks.

1. Introduction

It has been suggested to the writer by several colleagues (originally by Dr M. V. Barton) that the draining of fluid from a tank might have a significant effect on the damping of free-surface (sloshing) oscillations in the tank. Some of these colleagues also have conjectured that the incremental damping associated with decreasing depth might be negative, in analogy with the familiar situation for a pendulum of decreasing length (string being pulled through a small hole). We shall show here that linear theory predicts a positive damping that is small for all but very shallow tanks.

2. Direct solution

We consider a vertical cylinder of arbitrary cross-section S and horizontal bottom $z = 0$ that contains a prescribed, time-dependent volume of liquid $Sh(t)$, the undisturbed free surface being specified by $z = h(t)$. We then may specify the disturbed surface by

$$z = h(t) + \zeta(x, y, t), \tag{1}$$

where $\zeta(x, y, t)$ describes the sloshing motion; $|\zeta| \ll h$ by hypothesis. Let $\phi(x, y, z, t)$ be the velocity potential associated with this sloshing motion; then it is well known that appropriate linearized forms for ζ and ϕ are (Lamb 1932, § 259)

$$\zeta = q(t) f(x, y), \tag{2}$$

and

$$\phi = \dot{q}(t) f(x, y) \cosh(kz)/k \sinh(kh), \tag{3}$$

where $q(t)$ is a generalized co-ordinate for the modal distribution $f(x, y)$. This distribution must satisfy the Helmholtz equation,

$$\nabla^2 f + k^2 f = 0, \tag{4}$$

and the eigenvalue k is determined by the kinematic boundary condition $\phi_n = 0$ on the lateral walls of the cylinder.

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The linearized, free-surface condition is

$$\phi_t + \dot{h}\phi_z + g\zeta = 0 \quad (z = h(t)), \tag{5}$$

and differs from the conventional free-surface condition only in the addition of the term $\dot{h}\phi_z$, which represents the linearized perturbation of the term $\frac{1}{2}(\nabla\phi)^2$ in the pressure equation. Substituting (2) and (3) into (5) and dividing through by $k^{-1}\coth(kh)f$, we obtain the equation of motion

$$\ddot{q} - 2k\dot{h}\,\mathrm{cosech}\,(2kh)\,\dot{q} + gk\tanh(kh)\,q = 0. \tag{6}$$

3. Derivation from Hamilton's Principle

We also may proceed from (2) and (3) to (6) by way of Hamilton's principle [which implies (5)]. The kinetic energy of the sloshing motion is given by

$$T = \tfrac{1}{2}\rho\iint \phi\phi_n\,dS = \tfrac{1}{2}\rho\iint_S (\phi\phi_z)_{z=h}\,dS, \tag{7a,b}$$

where n denotes the outwardly directed normal on the boundary of the liquid; (7b) follows from (7a) by virtue of the boundary conditions $\phi_n = 0$ on the tank walls and $\phi_n = \phi_z$ at $z \doteqdot h$. The potential energy is given by

$$V = \tfrac{1}{2}\rho g\iint_S \zeta^2\,dS. \tag{8}$$

Substituting (2) and (3) into (7) and (8), we obtain the Lagrangian

$$T - V = \tfrac{1}{2}\rho[k^{-1}\coth(kh)\,\dot{q}^2 - gq^2]\iint_S f^2\,dS, \tag{9}$$

which yields the equation of motion

$$\frac{d}{dt}[\dot{q}\coth(kh)] + gkq = 0. \tag{10}$$

This is equivalent to (6).

4. Approximate solution

Let

$$y(t) = \tanh[kh(t)]. \tag{11}$$

We may obtain an asymptotic solution to (10), as $(gk)^{\frac{1}{2}}\dot{y} \to 0$, by the familiar method of Liouville. The first approximation is

$$q = y^{\frac{1}{4}}\exp\left[\pm i\int_{t_0}^{t}(gky)^{\frac{1}{2}}\,dt\right][1 + O(\dot{y}^2, \ddot{y})], \tag{12}$$

where the lower limit t_0 is an undetermined constant. We then may define the angular frequency ω and the logarithmic decrement according to

$$\int_t^{t+(2\pi/\omega)}(gky)^{\frac{1}{2}}\,dt = 2\pi, \tag{13}$$

and

$$\delta = \log\left[\frac{q(t)}{q(t+[2\pi/\omega])}\right]. \tag{14}$$

Substituting (12) into (13) and (14) and neglecting terms of order $\dot{y}^2$ and $\ddot{y}$, we obtain

$$\delta = -\tfrac{1}{2}\pi\dot{h}(gh)^{-\frac{1}{2}}(kh)^{\frac{1}{2}}[\sinh(kh)]^{-\frac{3}{2}}[\cosh(kh)]^{-\frac{1}{2}}, \tag{15}$$

and

$$\omega = [gk\tanh(kh)]^{\frac{1}{2}}(1-\delta). \tag{16}$$

We emphasize that the neglect of $\dot{y}^2$ and $\ddot{y}$ is not uniformly valid as $kh \to 0$.

In most cases of practical interest, we may approximate (15) by

$$\delta = -2\pi\dot{h}(gh)^{-\frac{1}{2}}(kh)^{\frac{1}{2}}e^{-2kh} \quad (kh \gg 1). \tag{17}$$

If we further assume that the drainage may be described by Torricelli's theorem according to

$$\dot{h} = -\alpha(2gh)^{\frac{1}{2}}, \tag{18}$$

where α is the total contraction ratio (of the *vena contracta* to πa^2), (17) becomes

$$\delta = 2\pi\alpha(2kh)^{\frac{1}{2}}e^{-2kh}. \tag{19}$$

This is likely to be even smaller than the viscous damping for most configurations. Consider, for example, a slowly draining circular cylinder for which $h \doteqdot a$ and $\alpha = 0{\cdot}05$; then $ka = 3{\cdot}84$ for the dominant mode, and $\delta \doteqdot 3 \times 10^{-4}$.

5. Experimental confirmation

Mr G. W. Brooks (National Aeronautics and Space Administration, Langley Field) has communicated some unpublished experimental data to the writer. Using a tank 36 in. in diameter and 60 in. mean depth draining through a 3·5 in. diameter centred hole at a rate varying from 0·89 to 1·08 in./sec, he found that the measured damping ratio did not differ significantly from that for the same configuration without drainage. These observations are consistent with the result (17).

We conclude that the slosh damping associated with drainage from a vertical tank is positive. It is also negligible for most practical configurations.

This work was carried out by the author for the Aerospace Corporation, El Segundo, California.

REFERENCE

Lamb, H. 1932 *Hydrodynamics*, 6th ed. Cambridge University Press.